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Darboux transformations for the Korteweg–de Vries equation

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Abstract. A Darboux transformation converting the Jost solution relating to the $(n-1)$-soliton solution of the KdV equation to that to the n-soliton solution is shown to be written in the form of a pole expansion and is then found explicitly for arbitrary n. Multisoliton solutions of the KdV equation are thus generated in practice by algebraic recursive procedures. The system of linear algebraic equations given by the inverse scattering method is reformulated, but without the restriction of $\lambda_1, \lambda_2, \ldots, \lambda_N$ in the upper half-plane of the spectral parameter λ. It is shown that an N-soliton solution specified by N couples of the constants $(\lambda_1, b_1), (\lambda_2, b_2), \ldots, (\lambda_N, b_N)$ is unaltered when one of these couples, say (λ_n, b_n), is transformed into $(\bar{\lambda}_n, b_n^{-1})$. Therefore, the usual restriction of $\lambda_1, \lambda_2, \ldots, \lambda_N$ is unnecessary and a definite N-soliton solution can be specified equally by 2^N possible sets of couples of the constants.

1. Introduction

The spectral significance of Bäcklund transformations in connection with nonlinear equations is that, typically, they have associated nonlinear superposition whereby multisoliton solutions to nonlinear equations may be generated by algebraic recursive procedures, in principle (Miura 1978, Rogers and Shadwick 1982). After discovering the inverse scattering method (Gardner *et al* 1967, Lax 1968, Zakharov and Shabat 1971, Ablowitz *et al* 1973), a transformation between two Jost solutions relating to the $(n-1)$- and n-soliton solutions of the same nonlinear equation is found to be more suitable for this purpose (Neugebauer and Meinel 1984, Levi *et al* 1984, Rangwala and Rao 1985, Gu and Zhou 1987). This may be called the Bäcklund transformation in the form of the Darboux matrix, or simply the Darboux transformation. However, to obtain explicit expressions of the Darboux transformations is still difficult, so that they were found merely for $n=1$ for the KdV equation and for equations of the ZS class.

Recently, due to introducing the Darboux transformation in pole expansion to replace the usual power expansion, explicit Darboux transformations have been found for arbitrary n for equations of the ZS class (Chen *et al* 1988, Xiao and Huang 1989). Afterwards, with the help of the so-called reduction transformation invariance, the same method gave explicit Darboux transformations for arbitrary n for the MNLS equation (Chen and Huang 1989), and for the DNLS equation (Huang and Chen 1990), both of which belong to the modified ZS class.

For the KdV equation, a particular restriction on Darboux transformations is given by the property that the 12 element of the matrix of the Lax pair is constant. To surmount this difficulty it has been suggested that the KdV equation of

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (1)$$

and that of

$$u_t - 6uu_x + u_{xxx} = 0 \tag{2}$$

must be simultaneously considered (Gu and Zhou 1987). In detail, a Darboux transformation is introduced to associate the Jost solution relating to the $(n-1)$-soliton solution of one of these equations and the n-soliton solution of the other. In this way, a system of equations which is compatible and can determine expressions of the Darboux transformations is derived. But, owing to complexity, an explicit expression of the Darboux transformation is obtained only in the case of $n = 1$ (Gu and Zhou 1987).

In this paper, extending the method based upon the Darboux transformation in pole expansion given in the paper of Chen *et al* (1988), with the same considerations of Gu and Zhou (1987), we show that the Darboux transformation is determined by another matrix which satisfies two differential equations. Though these two equations are still nonlinear we can transform them into the standard differential equations of a projection matrix so that their solution can be easily found in the usual way. The restriction given by the Lax pair mentioned above can be satisfied by choosing constants and suitably arranging two rows (or columns) of matrix factors involved in the general form of a projection matrix. Explicit Darboux transformations are thus obtained for arbitrary n for the KdV equation so that its multisoliton solutions can be generated in practice by algebraic recursive procedures.

Moreover, the system of linear algebraic equations given by the inverse scattering method is reformulated based upon these explicit Darboux transformations but without the restriction of $\lambda_1, \lambda_2, \ldots$ in the upper half-plane. The same situation has been met in the case of the NLS equation (Chen *et al* 1988).

In fact, as has been known for a long time, the usual restriction seems too harsh, since a two-soliton solution may be specified by two λ_j, $j = 1, 2$, one lying in the upper half-plane, the other in the lower half-plane, as seen from its explicit expression. On the other hand, in the Bäcklund transformation (Miura 1978, Rogers and Shadwick 1982), or in the direct method (Hirota 1971), there is no reason to give such a restriction. However, without any restriction, solutions may be irregular, e.g. a solution with λ_1 and λ_2 satisfying $\lambda_1 = \bar{\lambda}_2$, where the overbar denotes the complex conjugate. Therefore, if solutions are demanded to be regular, a certain relatively weak condition must be given to restrict these constants.

This problem has been solved for the NLS equation based upon the method of Darboux transformations in pole expansion (Huang and Liao 1991). In this paper, we have found explicit Darboux transformations for the KdV equation; we can show that an N-soliton solution specified by N couples of the constants (λ_j, b_j), $j = 1, 2, \ldots, N$ is unaltered when one of these couples, say (λ_n, b_n), is transformed into $(\bar{\lambda}_n, b_n^{-1})$. This result shows that the usual restriction given by the inverse scattering method is unnecessary and a definite N-soliton solution can be specified equally by 2^N possible sets of N couples of constants. At the same time this result shows that the usual restriction is allowable, since none of the known multisoliton solutions is ruled out.

2. Darboux transformations in pole expansion

The Lax pair of equations of (1) and (2) is

$$\partial_x \Phi(x, t, \lambda) = L(x, t, \lambda)\Phi(x, t, \lambda) \tag{3}$$

$$\partial_t \Phi(x, t, \lambda) = M(x, t, \lambda)\Phi(x, t, \lambda) \tag{4}$$

where

$$L(\lambda)=-\mathrm{i}\lambda\sigma_3+\begin{pmatrix}0 & -1\\ u & 0\end{pmatrix} \tag{5a}$$

$$M(\lambda)=-\mathrm{i}4\lambda^3\sigma_3+4\lambda^2\begin{pmatrix}0 & -1\\ u & 0\end{pmatrix}-\mathrm{i}2\lambda\begin{pmatrix}-u & 0\\ u_x & u\end{pmatrix}-\begin{pmatrix}-u_x & -2u\\ 2u^2+u_{xx} & u_x\end{pmatrix} \tag{5b}$$

for (1), and

$$L(\lambda)=-\mathrm{i}\lambda\sigma_3+\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix} \tag{6a}$$

$$M(\lambda)=-\mathrm{i}4\lambda^3\sigma_3+4\lambda^2\begin{pmatrix}0 & 1\\ u & 0\end{pmatrix}-\mathrm{i}2\lambda\begin{pmatrix}-u & 0\\ u_x & -u\end{pmatrix}+\begin{pmatrix}-u_x & 2u\\ 2u^2-u_{xx} & u_x\end{pmatrix} \tag{6b}$$

for (2). When soliton solutions are considered, a subscript n is added to specify quantities relating to the n-soliton solution. For definiteness, we suppose that u_n satisfies (1) or (2), according to whether n is even or odd.

Since

$$L_n(\lambda)\to-\mathrm{i}\lambda\sigma_3 \qquad M_n(\lambda)\to-\mathrm{i}4\lambda^3\sigma_3 \qquad \text{as } |\lambda|\to\infty \tag{7}$$

$\Phi_n(\lambda)$, $n=1,2,\ldots$ can be chosen to have the same asymptotic expressions in the limit as $|\lambda|\to\infty$. We then define Jost solutions $F_n(\lambda)$ recursively by Darboux transformations $D_n(\lambda)$ such that

$$F_n(\lambda)=D_n(\lambda)F_{n-1}(\lambda) \qquad n=1,2,\ldots \tag{8}$$

where

$$D_n(\lambda)\to I \qquad \text{as } |\lambda|\to\infty \tag{9}$$

and we assume

$$D_n(\lambda)=I+\frac{\lambda_n-\bar{\lambda}_n}{\lambda-\lambda_n}P_n \tag{10}$$

where the overbar denotes the complex conjugate and, as usual,

$$\lambda_n=\mathrm{i}k_n \tag{11}$$

where k_n is a real constant, and $P_n(x,t)$ is independent of λ. It is obvious that

$$F_0(\lambda)=\begin{pmatrix}\exp[-\mathrm{i}(\lambda x+4\lambda^3 t)] & -\exp[\mathrm{i}(\lambda x+4\lambda^3 t)]\\ 0 & \mathrm{i}2\lambda\exp[\mathrm{i}(\lambda x+4\lambda^3 t)]\end{pmatrix} \tag{12}$$

as seen from (3)-(5) with $u_0=0$. From (8)-(10) we can see that $F_n(\lambda)$ is composed of two Jost solutions with two components

$$F_n(\lambda)=(\tilde{\psi}_n(\lambda) \quad \tilde{\phi}_n(\lambda)) \tag{13}$$

which both have n single poles: $\lambda_1,\lambda_2,\ldots,\lambda_n$.

Similarly, we may consider

$$F'_n(\lambda)=D'_n(\lambda)F'_{n-1}(\lambda) \tag{14}$$

where

$$D'_n(\lambda) = I + \frac{\bar{\lambda}_n - \lambda_n}{\lambda - \bar{\lambda}_n} P'_n \tag{15}$$

$$F'_0(\lambda) = F_0(\lambda). \tag{16}$$

We thus have

$$F'_n(\lambda) = (\phi_n(\lambda) \quad \psi_n(\lambda)) \tag{17}$$

where $\phi_n(\lambda)$ and $\psi_n(\lambda)$ have n single poles: $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$.

From (3)-(6), we notice that each component of (13) and (17) satisfies the usual Lax pair of the KdV equation, e.g.

$$(-\partial_x^2 + (-1)^{n+1} u_n)\tilde{\psi}_n(\lambda)_1 = \lambda^2 \tilde{\psi}_n(\lambda)_1 \tag{18}$$

etc.

Substituting (8) into (3) and (4), we find

$$\partial_x D_n(\lambda) = L_n(\lambda) D_n(\lambda) - D_n(\lambda) L_{n-1}(\lambda) \tag{19}$$

$$\partial_t D_n(\lambda) = M_n(\lambda) D_n(\lambda) - D_n(\lambda) M_{n-1}(\lambda). \tag{20}$$

From (19) and (20) we have

$$U_n = U_{n-1} - \mathrm{i}(\lambda_n - \bar{\lambda}_n)[P_n, \sigma_3] \tag{21}$$

$$P_{nx} = L_n(\lambda_n) P_n - P_n L_{n-1}(\lambda_n) \tag{22}$$

$$P_{nt} = M_n(\lambda_n) P_n - P_n M_{n-1}(\lambda_n) \tag{23}$$

where

$$U_n = \begin{pmatrix} 0 & (-1)^{n+1} \\ u_n & 0 \end{pmatrix}. \tag{24}$$

Taking the limit as $\lambda \to \bar{\lambda}_n$ in (19) and (20) we obtain

$$-P_{nx} = L_n(\bar{\lambda}_n)(I - P_n) - (I - P_n) L_{n-1}(\bar{\lambda}_n) \tag{25}$$

$$-P_{nt} = M_n(\bar{\lambda}_n)(I - P_n) - (I - P_n) M_{n-1}(\bar{\lambda}_n). \tag{26}$$

From (22) and (25) we have

$$P_{nx} = L_{n-1}(\bar{\lambda}_n) P_n - P_n L_{n-1}(\lambda_n) + P_n(L_{n-1}(\lambda_n) - L_{n-1}(\bar{\lambda}_n)) P_n$$
$$+ (L_n(\lambda_n) - L_n(\bar{\lambda}_n))(P_n - P_n^2). \tag{27}$$

Similarly, from (23) and (26), we have

$$P_{nt} = M_{n-1}(\bar{\lambda}_n) P_n - P_n M_{n-1}(\lambda_n) + P_n(M_{n-1}(\lambda_n) - M_{n-1}(\bar{\lambda}_n)) P_n$$
$$+ (M_n(\lambda_n) - M_n(\bar{\lambda}_n))(P_n - P_n^2). \tag{28}$$

If P_n is a projection matrix

$$P_n^2 = P_n \tag{29}$$

the last terms on the right-hand sides of (27) and (28) vanish, and the remaining terms show that they are differential equations of a projection matrix. Therefore, (27) and (28) have a projection matrix solution.

From (21) we have

$$(P_n)_{12} = (-1)^{n-1} \frac{1}{\mathrm{i}(\lambda_n - \bar{\lambda}_n)} \tag{30}$$

$$u_n - u_{n-1} = -\mathrm{i}(\lambda_n - \bar{\lambda}_n) 2(P_n)_{21}. \tag{31}$$

Equation (30) gives a restriction on P_n to which we must carefully pay attention in solving (27) and (28).

From (29)-(31) P_n must be a real matrix:

$$\bar{P}_n = P_n. \tag{32}$$

This condition can be satisfied by taking pure imaginary $\lambda_1, \lambda_2, \ldots,$ as usual.

Similarly, from (14) and (15), we obtain two equations for P'_n, which have the same forms as (27) and (28), except for the interchange of λ_n and $\bar{\lambda}_n$. When P'_n is a projection matrix, the resultant equations are equivalent to those of $\sigma_2 P_n^{\mathrm{T}} \sigma_2$ obtained from (27) and (28), since

$$\sigma_2 L(\lambda)^{\mathrm{T}} \sigma_2 = -L(\lambda) \tag{33}$$

$$\sigma_2 M(\lambda)^{\mathrm{T}} \sigma_2 = -M(\lambda). \tag{34}$$

Therefore, we have

$$P'_n = \sigma_2 P_n^{\mathrm{T}} \sigma_2. \tag{35}$$

This formula can be also derived from other considerations.

3. An explicit solution for $n = 1$

Since $F_0(\lambda)$ has no singularity, we can set $\lambda = \lambda_1$ and $\lambda = \bar{\lambda}_1$. It is easily seen that a solution of (27) and (28) in the case of $n = 1$ is

$$P_1 = \frac{F_0(\bar{\lambda}_1)\begin{pmatrix} -b_1 \\ 1 \end{pmatrix}(1 \;\; b_1)\check{F}_0^{-1}(\lambda_1)}{(1 \;\; b_1)\check{F}_0^{-1}(\lambda_1) F_0(\bar{\lambda}_1)\begin{pmatrix} -b_1 \\ 1 \end{pmatrix}} \tag{36}$$

where b_1 is a real constant, and $\check{}$ indicates the interchange of two rows. From (12) we have

$$\check{F}_0^{-1}(\lambda) = \frac{1}{-\mathrm{i}2\lambda}\begin{pmatrix} 0 & -\exp[-\mathrm{i}(\lambda x + 4\lambda^3 t)] \\ -\mathrm{i}2\lambda \exp[\mathrm{i}(\lambda x + 4\lambda^3 t)] & -\exp[\mathrm{i}(\lambda x + 4\lambda^3 t)] \end{pmatrix}. \tag{37}$$

Setting $\lambda = \lambda_1$, $\lambda_1 = \mathrm{i}k_1$ every component of (12) and (37) becomes real. Explicitly, we have

$$(P_1)_{11} = \frac{b_1 \exp(-y)}{\exp(y) + b_1 \exp(-y)} \tag{38}$$

$$(P_1)_{12} = -\frac{1}{2k_1} \tag{39}$$

$$(P_1)_{22} = \frac{\exp(y)}{\exp(y) + b_1 \exp(-y)} \tag{40}$$

$$(P_1)_{21} = -\frac{2k_1 b_1}{\{\exp(y) + b_1 \exp(-y)\}^2} \tag{41}$$

where

$$y = k_1(x - 4k_1^2 t). \tag{42}$$

We thus see that (30) and (32) have been satisfied for $n = 1$.

From (31) we obtain

$$u_1 = -2k_1^2 \operatorname{sech}^2\{k_1[(x - x_1) - 4k_1^2 t]\} \tag{43}$$

where

$$x_1 = (2k_1)^{-1} \ln(b_1). \tag{44}$$

Equation (43) is the familiar expression of the one-soliton solution of the KdV equation of (2).

Substitution of (38), etc., yields

$$\tilde{\psi}_1(\lambda) = \begin{pmatrix} f_1(\lambda)_1 \\ f_1(\lambda)_2 \end{pmatrix} \tag{45}$$

$$\tilde{\phi}_1(\lambda) = \begin{pmatrix} \overline{g_1(\bar{\lambda})_1} \\ \overline{g_1(\bar{\lambda})_2 + \mathrm{i}2\lambda g_1(\bar{\lambda})_1} \end{pmatrix} \tag{46}$$

$$\phi_1(\lambda) = \begin{pmatrix} g_1(\lambda)_1 \\ g_1(\lambda)_2 \end{pmatrix} \tag{47}$$

$$\psi_1(\lambda) = \begin{pmatrix} \overline{f_1(\bar{\lambda})_1} \\ \overline{f_1(\bar{\lambda})_2 + \mathrm{i}2 f_1(\bar{\lambda})_1} \end{pmatrix} \tag{48}$$

where

$$f_1(\lambda)_1 = \left(1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1}(P_1)_{11}\right) \exp[-\mathrm{i}(\lambda x + 4\lambda^3 t)] \tag{49}$$

$$f_1(\lambda)_2 = \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1}(P_1)_{21} \exp[-\mathrm{i}(\lambda x + 4\lambda^3 t)] \tag{50}$$

$$g_1(\lambda)_1 = \left(1 + \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1}(P_1)_{22}\right) \exp[-\mathrm{i}(\lambda x + 4\lambda^3 t)] \tag{51}$$

$$g_1(\lambda)_2 = -\frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1}(P_1)_{21} \exp[-\mathrm{i}(\lambda x + 4\lambda^3 t)]. \tag{52}$$

From (8) we have

$$\phi_1(\lambda)_1 = \left[-\left(1 + \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1}(P_1)_{22}\right) - \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1}(P_1)_{12}\, \mathrm{i}2\lambda\right] \exp[\mathrm{i}(\lambda x + 4\lambda^3 t)]. \tag{53}$$

Because of (39), the term within large square brackets on the right-hand side of (53) becomes

$$1 + \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1}(P_1)_{11} = \overline{\left(1 + \frac{\lambda_1 - \bar{\lambda}_1}{\bar{\lambda} - \lambda_1}(P_1)_{11}\right)}. \tag{54}$$

We thus have

$$\psi_1(\lambda)_1 = \overline{f_1(\bar{\lambda})_1}. \tag{55}$$

Similarly, we can derive (46) and the second component of (48).

It is obvious that $f_1(\lambda)_1$, etc., and $\tilde{\psi}_1(\lambda)_1$, etc., are real when λ is pure imaginary. From (3), (4) and (6) we can see that (48) is a solution of them as long as (45) is. However, they have different poles: λ_1 for (45), $\bar{\lambda}_1$ for (48). We also have

$$\tilde{\phi}_1(\bar{\lambda}_1) = b_1\tilde{\psi}_1(\bar{\lambda}_1) \tag{56}$$

$$\phi_1(\lambda_1) = b_1\psi_1(\lambda_1) \tag{57}$$

$$\tilde{\psi}_1(\bar{\lambda}_1)_1 = f_1(\bar{\lambda}_1)_1 = \psi_1(\lambda_1)_1 \tag{58}$$

etc.

Since P_1 is a projection matrix, and noticing (15) and (35), we have

$$P_1 = I - \sigma_2 P_1^{\mathrm{T}}\sigma_2 = D_1'(\lambda_1). \tag{59}$$

We then have

$$F_1(\lambda) = \left(I + \frac{1}{\lambda - \lambda_1}\frac{1}{a_1}D_1'(\lambda_1)\right)F_0(\lambda) \tag{60}$$

where

$$a_1 = \frac{1}{\lambda_1 - \bar{\lambda}_1}. \tag{61}$$

Taking account of (57), we have

$$F_1(\lambda) = \left(I + \frac{1}{\lambda - \lambda_1}\frac{1}{a_1}\psi_1(\lambda_1)(b_1 \;\; 1)F_0^{-1}(\lambda_1)\right)F_0(\lambda). \tag{62}$$

Setting $\lambda = \bar{\lambda}_1$, noticing (56) and (58), the 11 element of the matrix of (62) is

$$\psi_1(\lambda_1)_1 = \left(1 + \frac{1}{\bar{\lambda}_1 - \lambda_1}\frac{b_1}{a_1}\psi_1(\lambda_1)_1 \exp[\mathrm{i}(\lambda_1 x + 4\lambda_1^3 t)]\right)\exp[-\mathrm{i}(\bar{\lambda}_1 x + 4\bar{\lambda}_1^3 t)]. \tag{63}$$

This equation is just the linear algebraic equation given by the inverse scattering method for the KdV equation of (2) for $n = 1$.

4. Explicit solutions for arbitrary n

We can obtain step by step all the necessary formulae for arbitrary n:

$$\tilde{\psi}_{n-1}(\lambda) = \begin{pmatrix} f_{n-1}(\lambda)_1 \\ f_{n-1}(\lambda)_2 \end{pmatrix} \tag{64}$$

$$\tilde{\phi}_{n-1}(\lambda) = \begin{pmatrix} (-1)^n\overline{g_{n-1}(\bar{\lambda})_1} \\ (-1)^n\overline{g_{n-1}(\bar{\lambda})_2 + \mathrm{i}2\lambda g_{n-1}(\bar{\lambda})_1} \end{pmatrix} \tag{65}$$

$$\phi_{n-1}(\lambda) = \begin{pmatrix} g_{n-1}(\lambda)_1 \\ g_{n-1}(\lambda)_2 \end{pmatrix} \tag{66}$$

$$\psi_{n-1}(\lambda) = \begin{pmatrix} (-1)^n\overline{f_{n-1}(\bar{\lambda})_1} \\ (-1)^n\overline{f_{n-1}(\bar{\lambda})_2 + \mathrm{i}2\lambda f_{n-1}(\bar{\lambda})_1} \end{pmatrix}. \tag{67}$$

These formulae are real when λ is pure imaginary. We have

$$P_n = \frac{F_{n-1}(\bar{\lambda}_n)\begin{pmatrix}-b_n\\1\end{pmatrix}(1\ \ b_n)\check{F}'^{-1}_{n-1}(\lambda_n)}{(1\ \ b_n)\check{F}'^{-1}_{n-1}(\lambda_n)F_{n-1}(\bar{\lambda}_n)\begin{pmatrix}-b_n\\1\end{pmatrix}} \tag{68}$$

where b_n is real constant, and λ_n is pure imaginary.

Explicitly, we have

$$(P_n)_{11} = (-1)^n \frac{b_n(-1)^{n-1}f_{n-1}(-\lambda_n)_2 + g_{n-1}(\lambda_n)_2 - i2\lambda_n b_n f_{n-1}(-\lambda_n)_1}{i2\lambda_n[b_n(-1)^{n-1}f_{n-1}(-\lambda_n)_1 + g_{n-1}(\lambda_n)_1]} \tag{69}$$

$$(P_n)_{12} = (-1)^{n-1}\frac{1}{i2\lambda_n} \tag{70}$$

$$(P_n)_{22} = (-1)^{n-1}\frac{b_n(-1)^{n-1}f_{n-1}(-\lambda_n)_2 + g_{n-1}(\lambda_n)_2 + (-1)^{n-1}\, i2\lambda_n g_{n-1}(\lambda_n)_1}{i2\lambda_n[b_n(-1)^{n-1}f_{n-1}(-\lambda_n)_1 + g_{n-1}(\lambda_n)_1]} \tag{71}$$

$$(P_n)_{21} = (-1)^n \frac{\begin{array}{c}(b_n(-1)^{n-1}f_{n-1}(-\lambda_n)_2 + g_{n-1}(\lambda_n)_2 - i2\lambda_n b_n f_{n-1}(-\lambda_n)_1)\\ \times (b_n(-1)^{n-1}f_{n-1}(-\lambda_n)_2 + g_{n-1}(\lambda_n)_2 + (-1)^{n-1}\, i2\lambda_n g_{n-1}(\lambda_n)_1)\end{array}}{i2\lambda_n[b_n(-1)^{n-1}f_{n-1}(-\lambda_n)_1 + g_{n-1}(\lambda_n)_1]^2}. \tag{72}$$

Since $\lambda_n = ik_n$, we have shown recursively that (30) and (32) are satisfied in the case of arbitrary n.

Substituting (64), etc., into (8), we obtain $F_n(\lambda)$ (which is (13)), each component of which has the same expressions as (64), etc., by replacing the suffix $n-1$ by n, respectively. From (31), we obtain the multisoliton solutions of the KdV equation by algebraic recursive procedures.

5. Reformulation of the system of linear algebraic equations given by the inverse scattering method

From the explicit expressions of the Darboux transformations we can obtain a system of linear algebraic equations which is the same as that given by the inverse scattering method but without the restriction of $\lambda_1, \lambda_2, \ldots,$ in the upper half-plane of the spectral parameter. From (8) and (10) we have

$$F_N(\lambda) = G_N(\lambda)F_0(\lambda) \tag{73}$$

where

$$G_N(\lambda) = D_N(\lambda)D_{N-1}(\lambda)\ldots D_1(\lambda). \tag{74}$$

Taking account of (10), by expanding (74) in the partial fraction, we obtain

$$G_N(\lambda) = I + \sum_{n=1}^{N}\frac{1}{\lambda-\lambda_n}A_n \tag{75}$$

where A_n is independent of λ, and

$$A_n = \lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)G_N(\lambda). \tag{76}$$

Since P_n are projection matrices, we have

$$I - P_n = \sigma_2 P_n^{\mathrm{T}} \sigma_2. \tag{77}$$

Taking account of (35), we have

$$D_n'(\lambda) = \frac{\lambda - \lambda_n}{\lambda - \bar{\lambda}_n} D_n(\lambda) \tag{78}$$

and then

$$G_N'(\lambda) = a(\lambda) G_N(\lambda) \tag{79}$$

where

$$G_N'(\lambda) = D_N'(\lambda) D_{N-1}'(\lambda) \ldots D_1'(\lambda) \tag{80}$$

$$a(\lambda) = \prod_{n=1}^{N} \frac{\lambda - \lambda_n}{\lambda - \bar{\lambda}_n}. \tag{81}$$

Taking the limit as $\lambda \to \lambda_n$, from (79) we obtain

$$G_N'(\lambda_n) = a_n \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n) G_N(\lambda) \tag{82}$$

where

$$a_n = \prod_{j=n} \frac{\lambda_n - \lambda_j}{\lambda_n - \bar{\lambda}_j} \frac{1}{\lambda_n - \bar{\lambda}_n}. \tag{83}$$

From (75), (76) and (82), we have

$$G_N(\lambda) = I + \sum_{n=1}^{N} \frac{1}{\lambda - \lambda_n} \frac{1}{a_n} G_N'(\lambda_n). \tag{84}$$

We now discuss the properties of

$$F_N(\bar{\lambda}_n) = (\tilde{\psi}(\bar{\lambda}_n) \quad \tilde{\phi}(\bar{\lambda}_n)) \tag{85}$$

and

$$F_N'(\lambda_n) = (\phi(\lambda_n) \quad \psi(\lambda_n)). \tag{86}$$

Here we have dropped the subscript N from the right-hand sides of (85) and (86). Noticing (35), we have

$$\begin{aligned} G_N'(\lambda_n) \check{F}_0(\lambda_n) &= \ldots P_n D_{n-1}'(\lambda_n) \ldots D_1'(\lambda_n) \check{F}_0(\lambda_n) \\ &= \ldots P_n \check{F}_{n-1}'(\lambda_n) \\ &= \ldots (1 \quad b_n) \end{aligned} \tag{87}$$

where the expression on the left of P_n on the right-hand side of (87) is not obviously written. We thus have

$$F_N'(\lambda_n) = \ldots (b_n \ \ 1) \tag{88}$$

and then

$$F_N'(\lambda_n) = \psi(\lambda_n)(b_n \ \ 1) \tag{89}$$

i.e.

$$\phi(\lambda_n) = b_n \psi(\lambda_n). \tag{90}$$

Similarly, we have

$$\begin{aligned} F_N(\bar{\lambda}_n) &= \dots (I - P_n) F_{n-1}(\bar{\lambda}_n) \\ &= \dots \sigma_2 P_n \sigma_2 F_{n-1}(\bar{\lambda}_n) \\ &= \dots (-b_n \ \ 1) F_{n-1}(\bar{\lambda}_n)^{\mathrm{T}} \sigma_2 F_{n-1}(\bar{\lambda}_n) \\ &= \dots (+b_n \ \ 1) \sigma_2 \\ &= \dots (1 \ \ b_n) \end{aligned} \tag{91}$$

and we thus have

$$F_N(\bar{\lambda}_n) = \tilde{\psi}(\bar{\lambda}_n)(1 \ \ b_n) \tag{92}$$

i.e.

$$\tilde{\phi}(\bar{\lambda}_n) = b_n \tilde{\psi}(\bar{\lambda}_n). \tag{93}$$

Substituting (89) into (84), we have

$$F_N(\lambda) = \left(I + \sum_{n=1}^{N} \frac{1}{\lambda - \lambda_n} \frac{1}{a_n} \psi(\lambda_n)(b_n \ \ 1) F_0^{-1}(\lambda_n) \right) F_0(\lambda). \tag{94}$$

The 11 element of the matrix of (94) is

$$\tilde{\psi}(\lambda)_1 = \left(1 + \sum_{n=1}^{N} \frac{1}{\lambda - \lambda_n} \frac{b_n}{a_n} \psi(\lambda_n)_1 \exp[\mathrm{i}(\lambda_n x + 4\lambda_n^3 t)] \right) \exp[-\mathrm{i}(\lambda x + 4\lambda^3 t)] \tag{95}$$

since

$$F_0^{-1}(\lambda) = \frac{1}{\mathrm{i}2\lambda} \begin{pmatrix} \mathrm{i}2\lambda \exp[\mathrm{i}(\lambda x + 4\lambda^3 t)] & \exp[\mathrm{i}(\lambda x + 4\lambda^3 t)] \\ 0 & \exp[-\mathrm{i}(\lambda x + 4\lambda^3 t)] \end{pmatrix}. \tag{96}$$

As $\tilde{\psi}(\lambda)_1$ satisfies (18) in setting $n = N$, we thus have

$$u_N(x, t) = (-1)^{N-1} \frac{2}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x} \left(\sum_{n=1}^{N} \frac{b_n}{a_n} \psi(\lambda_n)_1 \exp[\mathrm{i}(\lambda_n x + 4\lambda_n^3 t)] \right) \tag{97}$$

when considering the asymptotic behaviour of $\tilde{\psi}(\lambda)_1$ in the limit as $|\lambda| \to \infty$.

When $\lambda = \bar{\lambda}_m$, from (95) we have

$$\begin{aligned} (-1)^{N-1} & \psi(\lambda_m)_1 \\ &= \left(1 + \sum_{n=1}^{N} \frac{1}{\bar{\lambda}_m - \lambda_n} \frac{b_n}{a_n} \psi(\lambda_n)_1 \exp[\mathrm{i}(\lambda_n x + 4\lambda_n^3 t)] \right) \\ & \quad \times \exp[-\mathrm{i}(\bar{\lambda}_m x + 4\bar{\lambda}_m^3 t)] \end{aligned} \tag{98}$$

since

$$\tilde{\psi}(\bar{\lambda}_m)_1 = f_N(\bar{\lambda}_m)_1 = (-1)^{N-1} \psi(\lambda_m)_1 \tag{99}$$

which can be obtained from formulae similar to (64)–(67).

Equations (95), (97) and (98) are necessary formulae for giving the N-soliton solution of the KdV equation. These formulae are the same as those obtained by the inverse scattering method but without the restriction of $\lambda_1, \lambda_2, \dots, \lambda_N$ in the upper half-plane of λ.

6. Demonstration

We ought to show that the Jost solutions obtained by the above procedure satisfy the corresponding Lax equations.

P_n has been shown to be a projection matrix; we thus have

$$D_n^{-1}(\lambda)=I+\frac{\bar{\lambda}_n-\lambda_n}{\lambda-\bar{\lambda}_n}P_n \tag{100}$$

since

$$D_n(\lambda)D_n^{-1}(\lambda)=I \qquad \text{for arbitrary } \lambda. \tag{101}$$

We also have

$$G_N^{-1}(\lambda)=D_1^{-1}(\lambda)D_2^{-1}(\lambda)\dots D_N^{-1}(\lambda) \tag{102}$$

$$G'_N(\lambda)=\sigma_2 G_N^{-1}(\lambda)^{\mathrm{T}}\sigma_2. \tag{103}$$

Since $G_N^{-1}(\lambda)$ has N single poles: $\bar{\lambda}_1,\bar{\lambda}_2,\dots,\bar{\lambda}_N$, we write

$$F'^{-1}_N(\lambda)=F_0^{-1}(\lambda)G_N^{-1}(\lambda) \tag{104}$$

as mentioned below (14). From (101) we have

$$F_N(\lambda)F'^{-1}_N(\lambda)=I. \tag{105}$$

From (89) we have

$$\lim_{\lambda\to\lambda_n}(\partial_x F_N(\lambda))F'^{-1}_N(\lambda)$$

$$=\lim_{\lambda\to\lambda_n}\left(\frac{1}{\lambda-\lambda_n}\frac{1}{a_n}\psi_x(\lambda_n)(b_n\ \ 1)\right)\sigma_2(\psi(\lambda_n)(b_n\ \ 1))^{\mathrm{T}}\sigma_2=0. \tag{106}$$

Similarly, $(\partial_x F_N(\lambda))F'^{-1}_N(\lambda)$ vaniahes as $\lambda\to\bar{\lambda}_n$. Therefore, λ_n, $\bar{\lambda}_n$, $n=1,2,\dots,N$ are regular points of $(\partial_x F_N(\lambda))F'^{-1}_N(\lambda)$ and, similarly, of $(\partial_t F_N(\lambda))F'^{-1}_N(\lambda)$.

From (73) and (104) we have

$$(\partial_x F_N(\lambda))F'^{-1}_N(\lambda)=G_{Nx}(\lambda)G_N^{-1}(\lambda)+G_N(\lambda)L_0(\lambda)G_N^{-1}(\lambda). \tag{107}$$

$(\partial_x F_N(\lambda))F'^{-1}_N(\lambda)$ is thus analytic everywhere except at $\lambda=\infty$.

We expand $G_N(\lambda)$ into a Taylor series about $\lambda=\infty$:

$$G_N(\lambda)=\sum_{j=0}^{\infty}\alpha_j\lambda^{-j} \tag{108}$$

where

$$\alpha_0=I \tag{109}$$

$$\alpha_1=\sum_{n=1}^{N}A_n=\sum_{n=1}^{N}(\lambda_n-\bar{\lambda}_n)P_n \tag{110}$$

by (75) and (10). Therefore, we have

$$(\alpha_1)_{12}=-\mathrm{i}\text{ or }0 \qquad \text{for odd } N \text{ or even } N \tag{111}$$

$$(\alpha_1)_{21}=\mathrm{i}\tfrac{1}{2}u_N \tag{112}$$

because of (30) and (31).

Similarly, we have

$$G_N^{-1}(\lambda) = \sum_{k=0}^{\infty} \beta_k \lambda^{-k} \tag{113}$$

where

$$\beta_0 = I \qquad \beta_1 = -\alpha_1. \tag{114}$$

Substituting (108) and (113) into (107), we have

$$(\partial_x F_N(\lambda)) F_N'^{-1}(\lambda) = L_N(\lambda) + O(|\lambda|^{-1}) \tag{115}$$

since

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - i[\alpha_1, \sigma_3] = U_N. \tag{116}$$

Therefore, $(\partial_x F_N(\lambda)) F_N'^{-1}(\lambda) - L_N(\lambda)$ is analytic in the whole complex λ-plane and tends to zero as $|\lambda| \to \infty$; by the Liouville theorem it is equal to zero. This yields

$$G_{Nx}(\lambda) G_N^{-1}(\lambda) + G_N(\lambda) L_0(\lambda) G_N^{-1}(\lambda) = L_N(\lambda) \tag{117}$$

or

$$\partial_x F_N(\lambda) = L_N(\lambda) F_N(\lambda). \tag{118}$$

Similarly, we have

$$(\partial_t F_N(\lambda)) F_N'^{-1}(\lambda) = G_{Nt}(\lambda) G_N^{-1}(\lambda) + G_N(\lambda) M_0(\lambda) G_N^{-1}(\lambda). \tag{119}$$

$(\partial_t F_N(\lambda)) F_N'^{-1}(\lambda)$ is thus analytic everywhere except at $\lambda = \infty$. Owing to

$$M_0(\lambda) = 4\lambda^2 L_0(\lambda) \tag{120}$$

the right-hand side of (119) is

$$4\lambda^2(-i\lambda\sigma_3 + U_N) - 4\lambda\alpha_{1x} - 4(\alpha_{2x} + \alpha_{1x}\beta_1) + O(|\lambda|^{-1}). \tag{121}$$

From (117) we also have

$$G_{Nxx}(\lambda) + 2G_{Nx}(\lambda) L_0(\lambda) = (U_N^2 + U_{Nx}) G_N(\lambda). \tag{122}$$

Multiplying (122) by $\sigma_3 G_N^{-1}(\lambda)$ and expanding the resultant equation about $\lambda = \infty$ we obtain

$$2(-i)\alpha_{1x} = (U_N^2 + U_{Nx})\sigma_3 \tag{123}$$

$$\alpha_{1xx}\sigma_3 + 2(-i)(\alpha_{1x}\beta_1 + \alpha_{2x}) + 2\alpha_{1x} U_0 \sigma_3 = (U_N^2 + U_{Nx})(\alpha_1\sigma_3 + \sigma_3\beta_1). \tag{124}$$

From (123), we have

$$\alpha_{1x} = \frac{i}{2} \begin{pmatrix} (-1)^{N-1} u_N & 0 \\ u_{Nx} & -(-1)^{N-1} u_N \end{pmatrix}. \tag{125}$$

Taking account of (116), from (124) we have

$$\begin{aligned} \alpha_{1x}\beta_1 + \alpha_{2x} &= -\tfrac{1}{2}(U_N^2 + U_{Nx}) U_N + \tfrac{1}{4}(U_N^2 + U_{Nx})_x \\ &= \frac{1}{4} \begin{pmatrix} (-1)^{N-1} u_{Nx} & -2u_N \\ -(-1)^{N-1} 2u_N^2 + u_{Nxx} & -(-1)^{N-1} u_{Nx} \end{pmatrix}. \end{aligned} \tag{126}$$

Therefore, (121) is equal to $M_N(\lambda) + O(|\lambda|^{-1})$, and $(\partial_t F_N(\lambda)) F_N'^{-1}(\lambda) - M_N(\lambda)$ is analytic in the whole complex λ-plane and tends to zero as $|\lambda| \to \infty$. By the Liouville theorem it is equal to zero, and we thus obtain

$$\partial_t F_N(\lambda) = M_N(\lambda) F_N(\lambda). \tag{127}$$

7. Soliton solutions and locations of poles

Now we turn to the problem on the restriction on the poles of the transmission coefficient for specifying an N-soliton solution of the KdV equation given by the inverse scattering method (Gardner *et al* 1967). From (78) we have

$$D_n(\lambda)=\frac{\lambda-\bar{\lambda}_n}{\lambda-\lambda_n}D'_n(\lambda) \tag{128}$$

and then

$$F_n(\lambda)=\frac{\lambda-\bar{\lambda}_n}{\lambda-\lambda_n}\left(I+\frac{\bar{\lambda}_n-\lambda_n}{\lambda-\bar{\lambda}_n}\sigma_2P_n^{\mathrm{T}}\sigma_2\right)F_{n-1}(\lambda). \tag{129}$$

Substituting into (3) and (4), and eliminating the factor $(\lambda-\bar{\lambda}_n)/(\lambda-\lambda_n)$, we obtain

$$U_n=U_{n-1}-\mathrm{i}(\bar{\lambda}_n-\lambda_n)[\sigma_2P_n^{\mathrm{T}}\sigma_2,\sigma_3]. \tag{130}$$

Equations (21) and (130) are two different expressions for giving the same parameter, U_n. From (130) we have

$$u_n=u_{n-1}-\mathrm{i}(\bar{\lambda}_n-\lambda_n)2(\sigma_2P_n^{\mathrm{T}}\sigma_2)_{21}. \tag{131}$$

Equations (31) and (131) are, similarly, two different expressions for the same parameter, u_n.

It is easily seen that

$$\sigma_2\check{F}'^{-1}_{n-1}(\lambda_n)^{\mathrm{T}}\begin{pmatrix}1\\ b_n\end{pmatrix}=\prod_{j=1}^{n-1}\frac{\lambda_n-\lambda_j}{\lambda_n-\bar{\lambda}_j}b_nF'_{n-1}(\lambda_n)\begin{pmatrix}-b_n^{-1}\\ 1\end{pmatrix} \tag{132}$$

$$(-b_n\ \ 1)F_{n-1}(\bar{\lambda}_n)^{\mathrm{T}}\sigma_2=\prod_{j=1}^{n-1}\frac{\bar{\lambda}_n-\bar{\lambda}_j}{\bar{\lambda}_n-\lambda_j}b_n(1\ \ b_n^{-1})\check{F}^{-1}_{n-1}(\bar{\lambda}_n). \tag{133}$$

We thus have

$$\sigma_2P_n^{\mathrm{T}}\sigma_2=\frac{F'_{n-1}(\lambda_n)\begin{pmatrix}-b_n^{-1}\\ 1\end{pmatrix}(1\ \ b_n^{-1})\check{F}^{-1}_{n-1}(\bar{\lambda}_n)}{(1\ \ b_n)\check{F}^{-1}_{n-1}(\bar{\lambda}_n)F_{n-1}(\lambda_n)\begin{pmatrix}b_n^{-1}\\ 1\end{pmatrix}}. \tag{134}$$

It is necessary to write explicitly the couples of related constants from now on. From (134) we have

$$\sigma_2P_n(\lambda_n,b_n)^{\mathrm{T}}\sigma_2=P_n(\bar{\lambda}_n,b_n^{-1}). \tag{135}$$

From (129) we have

$$F_n(\lambda;\lambda_n,b_n)=\frac{\lambda-\bar{\lambda}_n}{\lambda-\lambda_n}F_n(\lambda;\bar{\lambda}_n,b_n^{-1}). \tag{136}$$

From (21) and (130) we obtain

$$U_n(\lambda_n,b_n)=U_n(\bar{\lambda}_n,b_n^{-1}) \tag{137}$$

where the couples of the constants $(\lambda_{n-1},b_{n-1}),\ldots,(\lambda_1,b_1)$ remain fixed.

When the couples of the constants $(\lambda_N, b_N), \ldots, (\lambda_{n+1}, b_{n+1})$ are unaltered, using (136) we have

$$P_{n+1}(\lambda_n, b_n) = P_{n+1}(\bar{\lambda}_n, b_n^{-1}) \tag{138}$$

$$F_{n+1}(\lambda; \lambda_n, b_n) = \frac{\lambda - \bar{\lambda}_n}{\lambda - \lambda_n} F_{n+1}(\lambda; \bar{\lambda}_n, b_n^{-1}) \tag{139}$$

$$U_{n+1}(\lambda_n, b_n) = U_{n+1}(\bar{\lambda}_n, b_n^{-1}) \tag{140}$$

where the unaltered couples of the constants are not written explicitly. Repeating the procedures step by step we finally obtain

$$P_N(\lambda_n, b_n) = P_N(\bar{\lambda}_n, b_n^{-1}) \tag{141}$$

$$F_N(\lambda; \lambda_n, b_n) = \frac{\lambda - \bar{\lambda}_n}{\lambda - \lambda_n} F_N(\lambda; \bar{\lambda}_n, b_n^{-1}) \tag{142}$$

$$U_N(\lambda_n, b_n) = U_N(\bar{\lambda}_n, b_n^{-1}). \tag{143}$$

In the above recursive formulae, P_n depends on quantities with subscripts less than n but not those with subscripts larger than n. However, the final results, (142) and (143), are independent of the order of these subscripts, since we have obtained a system of linear algebraic equations (98) which is symmetric with respect to these subscripts and gives directly $\psi_1(\lambda_N)_1$ and then u_N.

Equation (143) obviously yields that a definite N-soliton solution can be equally specified by 2^N possible sets of N couples of constants so that the usual restriction given by the inverse scattering method is unnecessary. However, this restriction is allowable, since all multisoliton solutions are included. it is easily seen that when λ_1, $\bar{\lambda}_1$, λ_2, $\bar{\lambda}_2, \ldots, \lambda_N$, $\bar{\lambda}_N$ are different from each other, the solution is regular and is indeed an N-soliton solution.

We have seen that the Darboux transformations for the KdV equation are more complicated than those for the NLS equation and for the sine-Gordon equation (Chen *et al* 1988, Xiao and Huang 1989). In the case of the NLS equation, it is known that the method of the Riemann problem with zeros (Zakharov and Shabat 1980) and the method of Darboux transformation in pole expansion are connected. The obtained expressions for Darboux transformations for the KdV equation may be of help in the Riemann problem with zeros in this case.

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